AFCRL-67-0299

PLANE WAVE PROPAGATION THROUGH A COMPRESSIBLE INHOMOGENEOUS PLASMA

BY

ERIC RAHNEBERG

GEORGE TYRAS

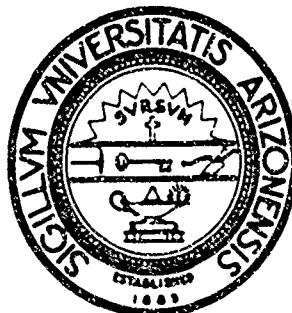THE UNIVERSITY OF ARIZONA
COLLEGE OF ENGINEERING
ENGINEERING EXPERIMENT STATION
TUCSON, ARIZONA

PREPARED FOR

D D C

JUN

31

PLANE WAVE PROPAGATION THROUGH A

COMPRESSIBLE INHOMOGENEOUS PLASMA

By

Eric Rahneberg

George Tyras

The University of Arizona
College of Engineering
Engineering Experiment Station
Tucson, Arizona

TABLE OF CONTENTS

ABSTRACT

The set of hydrodynamic equations together with Maxwell's equations which describe a nonuniform plasma as formulated by Cohen [1962] have been applied to a two dimensional problem where the magnetic field has only one component which is transverse to the direction of propagation and the profile is of the form e^{-Bx}. These assumptions lead to a set of second order coupled electromagnetic and hydrodynamic differential equations which describe the region under consideration. Solutions to this coupled set have been obtained by the application of the method of Frobenius. Considerable simplification resulted in the solutions when the assumption that the rms velocity of the electrons is much smaller than the speed of light was made. The resulting set of approximate solutions have been applied to the problem of obtaining reflection and transmission coefficients for the case of a plane wave incident from free space upon a layer of compressible inhomogeneous plasma.

LIST OF CONTRIBUTORS

RELATED CONTRACTS AND PUBLICATIONS

The present report is a continuation of research related to the plasma sheath effects on antennas. The previous publications are:

1. "An Experimental Study of Plasma Sheath Effects on Antennas" by G. Tyras, P. C. Bargeliotes, J. M. Hamm, and R. R. Schell, Scientific Report No. 1 AFCRL-65-53, Contract No. AF 19(628)-3834, December, 1964.

2. "An Experimental Study of Plasma Sheath Effects on Antennas" by G. Tyras, P. C. Bargeliotes, J. M. Hamm, and R. R. Schell, Radio Science, J. Res. NBS/VSNC-URSI, Vol. 69D, No. 6, June, 1965, pp. 839-850.

3. "Further Experimental Study of Plasma Sheath Effects on Antennas" by J. M. Hamm and G. Tyras, Scientific Report No. 2, AFCRL-65-608, Contract No. AF 19(628)-3834, July, 1965.

4. "Field of an Axially Slotted Circular Cylinder Clad with an Inhomogeneous Dielectric" by G. Tyras, Scientific Report No. 3, AF 19(628)-3834, February, 1966.

1. INTRODUCTION

While problems dealing with compressible homogeneous plasmas
and incompressible homogeneous and inhomogeneous plasmas have received
considerable attention in the past, little has appeared on the problem
of propagation through a plasma that is both compressible and inhomo-
geneous. Furthermore, one finds that frequent use is made of the bound-
ary condition that the normal component of the electron velocity shall
vanish at the free space-plasma interface. While the application of
this boundary condition may yield useable answers, it is doubtful that
rigid interfaces can be realized in situations dealing with reentry
communications. The boundary condition in question can be avoided by
suitably choosing a profile which is continuous at the air plasma inter-
face. The electron density will consequently be zero there and the
vanishing of the normal component of the electron velocity will be sat-
isfied automatically. Additional boundary conditions for the pressure
and its first derivative, however, must be introduced in order to specify
the problem uniquely.

In the following, the equations describing a compressible in-
homogeneous plasma will be derived first. Then a special case of the
latter will be considered and the corresponding solutions will be found.
Lastly, these will be applied to the determination of the reflection and
transmission coefficients for the case of a plane wave incident upon a
layer of compressible inhomogeneous plasma.

1

2. STATEMENT OF THE PROBLEM

Let us focus our attention on the problem of determining the transmission and reflection coefficients for the case of a plane wave incident obliquely upon a plasma layer which is both inhomogeneous and compressible. Let the layer be bounded by free space for the region x<0 on one side, and on the other, for x>t, by a compressible homogeneous plasma. The profile describing the inhomogeneity of the layer will be of the form exp(-Bx). The relative dielectric constant of free space will be taken as unity and that of the compressible homogeneous region as exp(-Bt). From this it follows that the relative dielectric constant is then continuous everywhere. Furthermore, it will be assumed that the magnetic field has only one component H_z and that the latter is transverse to the direction of propagation which has been chosen to be x. Finally, the problem is transformed into a two dimensional one by requiring that all variations along the z-axis to be zero. The reader is referred to Fig. 1, p.22.

Since the relevant equations for the regions x<0 and x>t are special cases of the equations describing the compressible inhomogeneous plasma, the latter region will be considered first.

3. DERIVATION OF THE EQUATIONS DESCRIBING AN INHOMOGENEOUS COMPRESSIBLE PLASMA

The linearized equations describing a warm, non-uniform, sourceless plasma as formulated by Cohen [1962] are

$$\nabla \times \vec{E} = i\omega\mu_o \vec{H} \tag{1}$$

$$\nabla \times \vec{H} = -i\omega\varepsilon_o \vec{E} - eN_o \vec{v} \tag{2}$$

$$\nabla \cdot \vec{H} = 0 \tag{3}$$

$$\nabla \cdot \vec{E} = \frac{-e}{\epsilon_o} N_1 \tag{4}$$

$$\nabla \cdot (N_o \vec{v}) = i\omega N_1 \tag{5}$$

$$-i\omega m N_o \vec{v} + N_o e \vec{E} = -\nabla P \tag{6}$$

$$P = \frac{N_o m v_o^2}{3} + N_1 m v_o^2 \tag{7}$$

where N_o and v_o are the average density and rms velocity of the electrons; m and e are the mass and charge of the electrons; $\vec{E}$ and $\vec{H}$ are the electro-magnetic field vectors; and $\vec{v}$ and N_1 are the velocity vector and density of the electrons imparted by the fields. In addition a time dependence of the form $e^{-i\omega t}$ has been assumed and suppressed throughout.

The velocity vector appearing in (2) may be eliminated by use of (6) yielding

$$\frac{\nabla \times \vec{H}}{\epsilon} = -i\omega \epsilon_o \vec{E} + \frac{ie}{\omega m} \frac{\nabla P}{\epsilon} \tag{8}$$

where

$$\epsilon = 1 - \frac{e^2 N_o(\vec{r})}{\epsilon_o \omega^2 m} . \tag{9}$$

Multiplying (8) by $\nabla \times$, using (1) and (3) and expanding produces

$$\nabla^2 \vec{H} + k_o^2 \epsilon \vec{H} + \frac{1}{\epsilon} \nabla \epsilon \times \nabla \times \vec{H} = \frac{ie}{\omega m} \frac{1}{\epsilon} \nabla \epsilon \times \nabla P \tag{10}$$

where

$$k_o^2 = \omega^2 \mu_o \epsilon_o .$$

4

Similarly the electric field vector appearing in (2) may be eliminated by the use of (6). Multiplication of the resultant equation by $\nabla\cdot$ produces

$$i\omega m\nabla\cdot(N_o\vec{v}) = \nabla\cdot(\frac{\nabla P}{\epsilon}) + \frac{ie}{\omega m}\nabla\cdot(\frac{N_o}{\epsilon}\nabla\times\vec{H}). \tag{11}$$

An expression for $\nabla\cdot(N_o\bar{v})$ may be obtained by solving (7) for N_1 and substituting the resultant expression into (5) producing

$$\nabla\cdot(N_o\vec{v}) = i\omega(\frac{P}{mv_o^2} - \frac{N_o}{3}). \tag{12}$$

The last term appearing in (12) is a static term and is henceforth dropped. Upon substituting of the time varying portion of (12) into (11) and expanding one obtains

$$\nabla^2 P + k_p^2\epsilon P - \frac{1}{\epsilon}\nabla\epsilon\cdot\nabla P = \frac{i\omega m}{e}\frac{1}{\epsilon}\nabla\epsilon\cdot\nabla\times H \tag{13}$$

where

$$k_p^2 = \frac{\omega^2}{v_o^2}. \tag{14}$$

An expression for the electric field may be obtained by solving (8) for $\vec{E}$. Then

$$\vec{E} = \frac{1}{\omega\epsilon_o\epsilon}\left[\nabla\times\vec{H} - \frac{ie}{\omega m}\nabla P\right] \tag{15}$$

Similarly, an expression for $\vec{v}$ may be derived by eliminating $\vec{E}$ from (2) and (6). Then

$$\vec{v} = \frac{-ie^2}{\omega^3\epsilon_o^2 m^2\epsilon(\epsilon-1)}\left[\nabla P + \frac{i\omega m}{e}(1-\epsilon)\nabla\times\vec{H}\right]. \tag{16}$$

Hence the plasma is completely specified once $\vec{H}$ and P defined by (10) and (13) are known.

4. SOLUTIONS OF THE COUPLED SYSTEM

Let us expand (10) and (13) in the cartesian coordinate system for the case where

$$\epsilon = \epsilon(x), \ \frac{\partial}{\partial z} = 0, \ \text{and} \ \vec{H} = \vec{\alpha}_z H_z \ .$$

Then

$$\frac{\partial^2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial y^2} - \frac{1}{\epsilon} \frac{\partial \epsilon}{\partial x} \frac{\partial H_z}{\partial x} + k_o^2 \epsilon(x) H_z = \frac{ie}{\omega m} \frac{1}{\epsilon} \frac{\partial \epsilon}{\partial x} \frac{\partial P}{\partial y} \tag{17}$$

and

$$\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} - \frac{1}{\epsilon} \frac{\partial \epsilon}{\partial x} \frac{\partial P}{\partial x} + k_p^2 \epsilon(x) P = i \frac{\omega m}{e} \frac{1}{\epsilon} \frac{\partial \epsilon}{\partial x} \frac{\partial H_z}{\partial y} \ . \tag{18}$$

In general the y-variation can be removed by a Fourier transform. However, since we are dealing with the propagation of plane waves through the plasma we can simply assume a y-variation of the form exp(iαy) where $\alpha = k_o \sin \theta_o$ and θ_o is the angle of incidence measured from the normal [Hessel et al 1962]. Equations (17) and (18) become

$$\left\{ \frac{d^2}{dx^2} - \frac{1}{\epsilon} \frac{d\epsilon}{dx} \frac{d}{dx} + k_o^2 \epsilon(x) - \alpha^2 \right\} H_z = \frac{-e\alpha}{\omega m} \frac{1}{\epsilon} \frac{d\epsilon}{dx} P$$

or in an abbreviated form

$$L_1 H_z = \frac{-e\alpha}{\omega m} \frac{1}{\epsilon} \frac{d\epsilon}{dx} P \tag{19}$$

6

$$\left\{ \frac{d^2}{dx^2} - \frac{1}{\epsilon} \frac{d\epsilon}{dx} \frac{d}{dx} + k_p^2 \epsilon(x) - \alpha^2 \right\} P = \frac{-\omega m}{e} \alpha \frac{1}{\epsilon} \frac{d\epsilon}{dx} H_z$$

$$L_2 P = \frac{-\omega m}{e} \alpha \frac{1}{\epsilon} \frac{d\epsilon}{dx} H_z \tag{20}$$

where L_1 and L_2 correspond to the differential operators in brackets.
Equations (19) and (20) are more readily solved if they can be decoupled.
Multiplying (19) by L_2 produces

$$L_2(L_1 H_z) \approx \frac{-e\alpha}{\omega m} L_2(\frac{1}{\epsilon} \frac{d\epsilon}{dx}) P \tag{21}$$

It is evident that P may be eliminated from (21) by means of (20) if L_2

and $\frac{1}{\epsilon} \frac{d\epsilon}{dx}$ commute. To meet this end, let us choose

$$\epsilon(x) = \exp(-Bx). \tag{22}$$

Then

$$L_2 L_1 H_z = \alpha^2 B^2 H_z \tag{23}$$

and similarly

$$L_1 L_2 P = \alpha^2 B^2 P . \tag{24}$$

Before proceeding with the solution of (23) and (24) let us return once
more to (19) and (20) to see if an additional simplification can be
effected. For the exponential profile (22), (19) and (20) become

$$\frac{d^2 H_z}{dx^2} + B \frac{dH_z}{dx} + (k_o^2 e^{-Bx} - \alpha^2) H_z = \alpha B \overline{P} \tag{25}$$

$$\frac{d^2\bar{P}}{dx^2} + B \frac{d\bar{P}}{dx} + (k_p^2 e^{-Bx} - \alpha^2)\bar{P} = \alpha B H_z \qquad (26)$$

where $\bar{P} = \frac{e}{\omega m} P$.

An equivalent system may be generated by the addition and subtraction of (25) and (26) which has the form

$$\frac{d^2 W}{dx^2} + B \frac{dW}{dx} + (\frac{k_p^2 + k_o^2}{2} e^{-Bx} - \alpha B - \alpha^2)W = \frac{k_p^2 - k_o^2}{2} e^{-Bx} R \qquad (27)$$

$$\frac{d^2 R}{dx^2} + B \frac{dR}{dx} + (\frac{k_p^2 + k_o^2}{2} e^{-Bx} + \alpha B - \alpha^2)R = \frac{k_p^2 - k_o^2}{2} e^{-Bx} W \qquad (28)$$

where

$$W = H_z + \bar{P}$$

and

$$R = H_z - \bar{P} .$$

$$(29)$$

Unlike Eqs. (25) and (26), (27) and (28) permit a comparison between the relative magnitudes of k_o^2 and k_p^2 . Let us consider the ratio

$$\frac{k_o^2}{k_p^2} = \frac{v_p^2}{c^2} \qquad (30)$$

where v_p is the r.m.s. velocity of electrons and c the velocity of light. Normally this ratio will be considerably smaller than unity, then to first order, any contribution due to k_o^2 may be neglected. The effect of neglecting k_o^2 in (27) and (28) is equivalent to dropping the term involving k_o^2 in (25). This result may easily be verified by the addition and subtraction of (27) and (28) and the subsequent use of (29) thus

8

recovering our original system (25) and (26) with the term involving k_o^2 absent. Thus we are faced with the task of solving

$$\left(\frac{d^2}{dx^2} + B\frac{d}{dx} - \alpha^2 \right) H_z = \frac{e}{\omega m} \alpha BP \tag{31}$$

and

$$\left(\frac{d^2}{dx^2} + B\frac{d}{dx} + k_p^2 e^{-Bx} - \alpha^2 \right) P = \frac{\omega m}{e} \alpha BH_z \tag{32}$$

By means of the transformation

$$P = e^{-\frac{Bx}{2}} F(x) \tag{33}$$

$$H_z = e^{-\frac{Bx}{2}} G(x) \tag{34}$$

(31) and (32) become

$$\frac{d^2 G}{dx^2} - (\alpha^2 + \frac{B^2}{4})G = \frac{e}{\omega m} \alpha BF \tag{35}$$

and

$$\frac{d^2 F}{dx^2} + (k_p^2 e^{-Bx} - \alpha^2 - \frac{B^2}{4})F = \frac{\omega m}{e} \alpha BG , \tag{36}$$

Equations (35) and (36) may be decoupled according to the method indicated by (23) and (24). Then

$$\frac{d^4 F}{dx^4} + \left(k_p^2 e^{-Bx} - 2(\alpha^2 + \frac{B^2}{4}) \right) \frac{d^2 F}{dx^2} - 2B\, k_p^2 e^{-Bx} \frac{dF}{dx}$$

$$+ \left((\alpha^2 - \frac{B^2}{4})^2 + k_p^2 (\frac{3B^2}{4} - \alpha^2)e^{-Bx} \right) F = 0 \tag{37}$$

$$\frac{d^4 G}{dx^4} + \left[k_p^2 e^{-Bx} - 2(\alpha^2 + \frac{B^2}{4}) \right] \frac{d^2 G}{dx^2}$$

$$+ \left[(\alpha^2 - \frac{B^2}{4})^2 - k_p^2(\alpha^2 + \frac{B^2}{4})e^{-Bx} \right] G = 0 . \tag{38}$$

Before applying the method of Frobenius to each of the decoupled equations it is convenient to remove the factor e^{-Bx}. To this end let

$$F(x) = f(z)$$

$$G(x) = g(z) \tag{39}$$

where $z = \exp(\frac{-Bx}{2})$.

Then

$$z^4 \frac{d^4 f}{dz^4} + 6z^3 \frac{d^3 f}{dz^3} + z^2 \left[Kz^2 + 7 - 2v^2 \right] \frac{d^2 f}{dz^2}$$

$$+ z \left[5Kz^2 + 1 - 2v^2 \right] \frac{df}{dz} + \left[K(4 - v^2)z^2 + v^4 - \gamma^2 \right] f = 0 \tag{40}$$

and

$$z^4 \frac{d^4 g}{dz^4} + 6z^3 \frac{d^3 g}{dz^3} + z^2 \left[Kz^2 + 7 - 2v^2 \right] \frac{d^2 g}{dz^2}$$

$$+ z \left[Kz^2 + 1 - 2v^2 \right] \frac{dg}{dz} + \left[v^4 - \gamma^2 - Kv^2 z^2 \right] g = 0 \tag{41}$$

where

$$K = \frac{4k_p^2}{B^2}, \quad v^2 = \frac{4\alpha^2}{B^2} + 1, \quad \text{and} \quad \gamma^2 = \frac{16\alpha^2}{B^2} .$$

10

Assuming power series solutions of the form

$$f(z) = \sum_{n=o}^{\infty} a_n z^{n+s}$$

$$g(z) = \sum_{n=o}^{\infty} b_n z^{n+s}$$

(42)

one obtains upon substitution into (40) and (41) the following recursion relationships

$$a_n = - \frac{4k_p^2}{B^2} \frac{[(n+s)^2 - v^2]}{[(n+s)^2 - v^2]^2 - \gamma^2} a_{n-2}$$

$$b_n = - \frac{4k_p^2}{B^2} \frac{[(n-2+s)^2 - v^2]}{[(n+s)^2 - v^2]^2 - \gamma^2} b_{n-2}$$

(43)

where s is determined from the indicial equation

$$(s^2 - v^2)^2 - \gamma^2 = 0 .$$

(44)

Since two of the roots of the indicial equation are also zeros of the denominators of the recursion relations for the case n = 2, both solutions of the first and second kind must be constructed. As this procedure is rather well-known, only the final results will be presented. Thus the solutions generated by the method of Forbenius are

$$P_A(x) = e^{-x(\alpha+B)} \left(1 + \sum_{m=1}^{\infty} (-1)^m \frac{\Delta(1)\Delta(2)\ldots\Delta(m)}{D(1)D(2)\ldots D(m)} \left(\frac{k_p^2}{B^2} e^{-Bx} \right)^m \right)$$

(45a)

$$H_A(x) = e^{-x(\alpha+B)}\left[1 + \sum_{m=1}^{\infty} (-1)^m \frac{\Delta(0)\Delta(1)\ldots\Delta(m-1)}{D(1)D(2)\ldots D(m)}\left(\frac{k_p^2}{B^2} e^{-Bx}\right)^m\right]$$

(45b)

$$P_B(x) = P_A(x,-\alpha)$$ (45c)

$$H_B(x) = H_A(x,-\alpha)$$ (45d)

$$P_C(x) = e^{-\alpha x}(\frac{-Bx}{2}) \frac{1}{\frac{\alpha}{B}(1+\frac{2\alpha}{B})} \sum_{m=1}^{\infty} (-1)^m \frac{\Delta(0)\Delta(1)\ldots\Delta(m-1)}{D(1)D(2)\ldots D(m-1)}\left(\frac{k_p^2}{B^2} e^{-Bx}\right)^m$$

$$+ e^{-\alpha x}\left[1 + \frac{1}{\frac{\alpha}{B}(1+\frac{2\alpha}{B})} \sum_{m=1}^{\infty} (-1)^m \frac{\Delta(0)\Delta(1)\ldots\Delta(m-1)}{D(1)D(2)\ldots D(m-1)}\left(\frac{B^2+2\alpha B+4\alpha^2}{4\alpha(B+2\alpha)}\right.\right.$$

$$+ \sum_{k=2}^{m}\left(\frac{\Delta'(k)}{2\Delta(k-1)} - \frac{\Delta'(k)\Delta(k-1)}{D(k-1)}\right)\right)\left(\frac{k_p^2}{B^2} e^{-Bx}\right)^m\right]$$ (45e)

$$H_C(x) = e^{-\alpha x}(\frac{-Bx}{2}) \frac{1}{\frac{\alpha}{B}(1+\frac{2\alpha}{B})} \sum_{m=1}^{\infty} (-1)^m \frac{\Delta(-1)\Delta(0)\ldots\Delta(m-2)}{D(1)D(2)\ldots D(m-1)}\left(\frac{k_p^2}{B^2} e^{-Bx}\right)^m$$

$$+ e^{-\alpha x}\left[1 + \frac{1}{\frac{\alpha}{B}(1+\frac{2\alpha}{B})} \sum_{m=1}^{\infty} (-1)^m \frac{\Delta(-1)\Delta(0)\ldots\Delta(m-2)}{D(1)D(2)\ldots D(m-1)}\left(\frac{B^2-6\alpha B-12\alpha^2}{4\alpha(B+2\alpha)}\right.\right.$$

$$+ \sum_{k=2}^{m}\left(\frac{\Delta'(k-1)}{2\Delta(k-2)} - \frac{\Delta'(k)\Delta(k-1)'}{D(k-1)}\right)\right)\left(\frac{k_p^2}{B^2} e^{-Bx}\right)^m\right]$$ (45f)

12

$$P_D(x) = P_C(x, -\alpha) \tag{45g}$$

$$H_D(x) = H_C(x, -\alpha) \tag{45h}$$

where

$$\Delta(m) = m(m+1) + \frac{\alpha}{B}(2m+1)$$

$$D(m) = m(m+1)\ (m + \frac{2\alpha}{B})\ (m + 1 + \frac{2\alpha}{B}) \tag{45i}$$

$$\Delta'(m) = (2m + \frac{2\alpha}{B} - 1)\ .$$

The above solutions hold for the case where $\frac{2\alpha}{B}$ is not equal to an integer or zero.

Substituting the generated series into (31) and (32) reveals only P_A, $\frac{e}{\omega m} H_A$ and P_B, $\frac{-e}{\omega m} H_B$ to be solutions. Additional solutions, however, may be found by forming linear superpositions of the various series generated. Thus, if

$$H_3(x) = \frac{e}{\omega m} \gamma_A\ H_A(x) + \frac{e}{\omega m} \gamma_B\ H_C(x) \tag{46}$$

$$P_3(x) = \gamma_C\ P_A(x) + \gamma_D P_C(x) \tag{47}$$

then a substitution of (46) and (47) into (31) and (32) reveals that

$$\gamma_C = \gamma_A + \frac{k_p^{\ 2}(B - 2\alpha)}{2\alpha B(B + 2\alpha)}\ \gamma_B \tag{48}$$

$$\gamma_D = -\ \gamma_B\ . \tag{49}$$

Hence, the various γ's cannot be determined uniquely. Substitution of (48) and (49) into (46) and (47) produces

$$P_3 = \gamma_B\ \left(\frac{k_p^{\ 2}(B-2\alpha)}{2\alpha B(B+2\alpha)}\ P_A - P_C \right)\ +\ \gamma_A P_A \tag{50}$$

$$H_3 = \frac{e}{\omega m}\ \gamma_B H_C + \frac{e}{\omega m}\ \gamma_A H_A \tag{51}$$

or alternately

$$P_3 = - \gamma_B P_C + \gamma_C P_A \tag{52}$$

$$H_3 = \frac{e}{\omega m} \gamma_B \left(H_C - \frac{k_p^2 (B-2\alpha)}{2\alpha B (B+2\alpha)} H_A \right) + \frac{e}{\omega m} \gamma_C H_A \tag{53}$$

The last terms in (50) to (53) may be dropped, since being already
solutions of the system, they add nothing new. That the alternate forms
are equivalent and that either representation may be used for the prob-
lem at hand will be demonstrated shortly. Similarly, if

$$n_4(x) = - \frac{e}{\omega m} \gamma_G H_B(x) - \frac{e}{\omega m} \gamma_H h_D(x) \tag{54}$$

$$P_4(x) = \gamma_E P_B + \gamma_F P_D \tag{55}$$

then

$$\gamma_E = \gamma_G + \frac{k_p^2 (B+2\alpha)}{-2\alpha B (B-2\alpha)} \gamma_H \tag{56}$$

$$\gamma_H = - \gamma_F \tag{57}$$

and

$$P_4 = \gamma_H \left(\frac{k_p^2 (B + 2\alpha)}{-2\alpha B (B-2\alpha)} P_B - P_D \right) + \gamma_G P_B \tag{58}$$

$$H_4 = - \gamma_H \frac{e}{\omega m} H_D - \frac{e}{\omega m} \gamma_B H_B \tag{59}$$

or

$$P_4 = - \gamma_H P_D + \gamma_E P_B \tag{60}$$

14

$$H_4 = \frac{e}{\omega m} \left[\frac{k_p^2 (B + 2\alpha)}{-2\alpha B(B-2\alpha)} H_B - H_D \right] - \gamma_E \frac{e}{\omega m} H_B \qquad (61$$

Again the last terms appearing in (59) through (61 may be dropped since they are already solutions of the coupled system.

The sets of solutions satisfying (31) and (32) are summarized below

$$P_1(x) = P_A(x)$$

$$H_1(x) = \frac{e}{\omega m} H_A(x)$$

$$P_2(x) = P_B(x) \qquad (62)$$

$$H_2(x) = -\frac{e}{\omega m} H_B(x)$$

$$P_3(x) = \frac{k_p^2 (B-2\alpha)}{2\alpha B(B+2\alpha)} P_A(x) - P_c(x)$$

$$H_3(x) = \frac{e}{\omega m} H_c(x)$$

$$P_4(x) = \frac{k_p^2 (B+2\alpha)}{-2\alpha B(B-2\alpha)} P_B(x) - P_D(x)$$

$$H_4(x) = -\frac{e}{\omega m} H_D(x) .$$

The complete solution describing the pressure and magnetic field for the region $0 \le x \le t$ is obtained by forming a linear superposition of all the solutions tabulated above. Thus

$$H(x) = C_1 H_1(x) + C_2 H_2(x) + C_3 H_3(x) + C_4 H_4(x) \qquad (63)$$

and

$$P(x) = C_1 P_1(x) + C_1 P_1(x) + C_1 P_1(x) + C_1 P_1(x)$$

Recalling that the C's are yet unspecified the substitution

$$C_1 = C_1' - \frac{k_p^{\ 2}(B-2\alpha)}{2\alpha B(B+2\alpha)} \tag{65}$$

$$C_2 = C_2' - \frac{k_p^{\ 2}(B+2\alpha)}{-2\alpha B(B-2\alpha)} \tag{66}$$

transforms (65) and (66) into the form which would have resulted if the alternate forms for P_3, P_4, H_3 and H_4 had been used. Thus it can be concluded that the various forms are equivalent.

5. THE BOUNDARY CONDITIONS

Having obtained solutions to the coupled set of electromagnetic and hydrodynamic equations a consideration of the boundary conditions at the interfaces located at $x = 0$ and $x = t$ is in order. For the electro-magnetic fields we shall require that the tangential components of the electric and magnetic fields are continuous at the interfaces. In order to determine the behavior of P at the free space-plasma interface Eq. (6) is expanded in component form. Thus

$$-\frac{\partial P}{\partial y} = N_o e\, E_y - i\omega m\, N_o v_y \tag{67}$$

and

$$-\frac{\partial P}{\partial x} = N_o e\, E_x - i\omega m\, N_o v_x \tag{68}$$

where $N_o = \dfrac{\epsilon_o m\omega^2}{e^2}(1 - e^{-Bx})$.

Since the electron density N_o is equal to zero for x equal to zero and the components of the electric field and ion velocity are expected to be

16

bounded it follows that

$$\frac{\partial P}{\partial x}\bigg|_{x=o+} = 0 \tag{69}$$

$$P\bigg|_{x=o+} = 0 \tag{70}$$

In addition we shall require that P be continuous at $x = t$.

6. THE DETERMINATION OF THE REFLECTION AND TRANSMISSION COEFFICIENTS

The partial differential equation describing the magnetic field in free space is obtained from (15) for the case $\varepsilon = 1$. The solutions are well-known and the expression for the magnetic field may be written as

$$H_{z1} = (e^{ik_o\cos\theta_o x} + Re^{-ik_o\cos\theta_o x}) \, e^{ik_o\sin\theta_o y} \tag{71}$$

where R is the reflection coefficient sought and θ_o the angle of incidence measured from the normal. Using (15) it follows that the expression for the tangential component of the electric field is

$$E_{y1} = \frac{k_o\cos\theta_o}{\omega\varepsilon_o}(e^{ik_o\cos\theta_o x} - Re^{-ik_o\cos\theta_o x}) \, e^{ik_o\sin\theta_o y} . \tag{72}$$

For the region occupied by the compressible inhomogeneous plasma we have

$$H_{z2} = \left[C_1 H_1(x) + C_2 H_2(x) + C_3 H_3(x) + C_4 H_4(x)\right] e^{ik_o\sin\theta_o y} \tag{73}$$

$$P = \left[C_1 P_1(x) + C_2 P_2(x) + C_3 P_3(x) + C_4 P_4(x)\right] e^{ik_o\sin\theta_o y} . \tag{74}$$

The tangential component of the electric field E_{y2} may be calculated

from (15). By making use of (73) and (74) this may be expressed as

$$E_{y2} = \frac{-1}{\omega\epsilon_o\epsilon(x)}\left[C_1(H_1' - \frac{e\alpha}{\omega m} P_1) + C_2(H_2' - \frac{e\alpha}{\omega m} P_2) + C_3(H_3' - \frac{e\alpha}{\omega m} P_3)\right.$$

$$\left. + C_4(H_4' - \frac{e\alpha}{\omega m} P_4)\right] \tag{75}$$

where the prime denotes differentiation with respect to x and $\alpha = k_o\sin\theta_o$.

The equations describing the pressure and the magnetic field for the region occupied by the homogeneous compressible plasma may be obtained from (17) and (18) setting $\epsilon = \epsilon_3 = \exp(-Bt) = $ constant. Then

$$\frac{d^2H_{z3}}{dx^2} + (k_o^2\epsilon_3 - \alpha^2)H_{z3} = 0 \tag{76}$$

$$\frac{d^2P_3}{dx^2} + (k_p^2\epsilon_3 - \alpha^2)P_3 = 0 . \tag{77}$$

Since the region for x > t is semi-infinite only outgoing waves will be present. Thus solutions for this region are of the form

$$H_{z3} = Te^{+i\sqrt{k_o^2\epsilon_3 - \alpha^2} (x-t)} e^{ik_o\sin\theta_o y} \tag{78}$$

$$P_3 = Qe^{+i\sqrt{k_p^2\epsilon_3 - \alpha^2} (x-t)} e^{ik_o\sin\theta_o y} . \tag{79}$$

As before the tangential component of the electric field E_{y3} is obtained from (15). Making use of (78) and (79)

$$E_{y3} = \left[\frac{\sqrt{k_o^2\epsilon_3 - \alpha^2}}{\omega\epsilon_o\epsilon_3} Te^{i\sqrt{k_o^2\epsilon_3 - \alpha^2} (x-t)}\right.$$

$$\left. + \frac{ie\alpha}{\omega^2 m\epsilon_o\epsilon_3} Qe^{i\sqrt{k_p^2\epsilon_3 - \alpha^2} (x-t)}\right]e^{ik_o\sin\theta_o y} . \tag{80}$$

$$\begin{pmatrix}
H_1(0) & H_2(0) & H_3(0) & H_4(0) & -1 & 0 & 0 \\
H_1'(0) & H_2'(0) & H_3'(0) & H_4'(0) & ik_0\cos\theta_0 & 0 & 0 \\
P_1(0) & P_2(0) & P_3(0) & P_4(0) & 0 & 0 & 0 \\
P_1'(0) & P_2'(0) & P_3'(0) & P_4'(0) & 0 & 0 & 0 \\
H_1(t) & H_2(t) & H_3(t) & H_4(t) & 0 & -1 & 0 \\
P_1(t) & P_2(t) & P_3(t) & P_4(t) & 0 & 0 & -1 \\
H_1'(t)-\dfrac{e\alpha}{\omega m}P_1(t) & H_2'(t)-\dfrac{e\alpha}{\omega m}P_2(t) & H_3'(t)-\dfrac{e\alpha}{\omega m}P_3(t) & H_4'(t)-\dfrac{e\alpha}{\omega m}P_4(t) & 0 & -i\sqrt{k_0^2\epsilon_3-\alpha^2} & \dfrac{e\alpha}{\omega m}
\end{pmatrix}
\begin{pmatrix}
C_1 \\ C_2 \\ C_3 \\ C_4 \\ R \\ T \\ Q
\end{pmatrix}
=
\begin{pmatrix}
1 \\ ik_0\cos\theta_0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0
\end{pmatrix}$$

(81)

Application of the indicated boundary conditions yields (81) where the desired coefficients may be determined by applying Cramers rule.

7. NUMERICAL RESULTS

For purposes of numerical evaluation the reflection and transmission coefficients may be expressed as

$$R = -\frac{W_1 - k_o\cos\theta_o\sqrt{k_o^2\epsilon_3 - \alpha^2}\,W_4 - i(k_o\cos\theta_o W_3 + \sqrt{k_o^2\epsilon_3 - \alpha^2}\,W_2)}{W_1 + k_o\cos\theta_o\sqrt{k_o^2\epsilon_3 - \alpha^2}\,W_4 + i(k_o\cos\theta_o W_3 - \sqrt{k_o^2\epsilon_3 - \alpha^2}\,W_2)}$$

(82)

$$T = \frac{i2k_o\cos\theta_o W_5}{W_1 + k_o\cos{}_o\sqrt{k_o^2\epsilon_3 - \alpha^2}\,W_4 + i(k_o\cos\theta_o W_3 - \sqrt{k_o^2\epsilon_3 - \alpha^2}\,W_2)}$$

(83)

where

$$W_1 = \begin{vmatrix} H_1'(o) & H_2'(o) & H_2'(o) & H_4'(o) & 0 \\ P_1(o) & P_2(o) & P_3(o) & P_4(o) & 0 \\ P_1'(o) & P_2'(o) & P_3'(o) & P_4'(o) & 0 \\ P_1(t) & P_2(t) & P_3(o) & P_4(t) & -1 \\ H_1'(t) - \frac{e\alpha}{\omega m}P_1(t) & H_2'(t) - \frac{e\alpha}{\omega m}P_2(t) & H_3'(t) - \frac{e\alpha}{\omega m}P_3(t) & H_4'(t) - \frac{e\alpha}{\omega m}P_4(t) & \frac{e\alpha}{\omega m} \end{vmatrix}$$

(84)

$$W_2 = \begin{vmatrix} H_1'(o) & H_2'(o) & H_3'(o) & H_4'(o) \\ P_1(o) & P_2(o) & P_3(o) & P_4(o) \\ P_1'(o) & P_2'(o) & P_3'(o) & P_4'(o) \\ H_1(t) & H_2(t) & H_3(t) & H_4(t) \end{vmatrix}$$

(85)

$$W_3 = \begin{vmatrix} H_1(o) & H_2(o) & H_3(o) & H_4(o) & 0 \\ P_1(o) & P_2(o) & P_3(o) & P_4(o) & 0 \\ P_1'(o) & P_2'(o) & P_3'(o) & P_4'(o) & 0 \\ P_1(t) & P_2(t) & P_3(t) & P_4(t) & -1 \\ H_1'(t)-\frac{e\alpha}{\omega m}P_1(t) & H_2'(t)-\frac{e\alpha}{\omega m}P_2(t) & H_3'(t)-\frac{e\alpha}{\omega m}P_3(t) & H_4'(t)-\frac{e\alpha}{\omega m}P_4(t) & \frac{e\alpha}{\omega m} \end{vmatrix}$$

(86)

$$W_4 = \begin{vmatrix} H_1(o) & H_2(o) & H_3(o) & H_4(o) \\ P_1(o) & P_2(o) & P_3(o) & P_4(o) \\ P_1'(o) & P_2'(o) & P_3'(o) & P_4'(o) \\ H_1(t) & H_2(t) & H_3(t) & H_4(t) \end{vmatrix}$$

(87)

$$W_5 = \begin{vmatrix} P_1(o) & P_2(o) & P_3(o) & P_4(o) & 0 \\ P_1'(o) & P_2(o) & P_3'(o) & P_4'(o) & 0 \\ H_1(t) & H_2(t) & H_3(t) & H_4(t) & 0 \\ P_1(t) & P_2(t) & P_3(t) & P_4(t) & -1 \\ H_1'(t)-\frac{e\alpha}{\omega m}P_1(t) & H_2'(t)-\frac{e\alpha}{\omega m}P_2(t) & H_3'(t)-\frac{e\alpha}{\omega m}P_3(t) & H_4'(t)-\frac{e\alpha}{\omega m}P_4(t) & \frac{e\alpha}{\omega m} \end{vmatrix}$$

(88)

It should be noted that all of the entries in the above determinants are
real. Furthermore, expressions for the reflection and transmission coef-
ficients for the case where $k_o^2 \epsilon_3 - \alpha^2 < o$ may be obtained by replacing
the factor $(k_o^2 \epsilon_3 - \alpha^2)^{1/2}$ appearing in (82) and (83) by $+i(\alpha^2 - k_o\epsilon_3)^{1/2}$

While all of the series solutions converge, their rate of conver-
gence depends critically upon the magnitude of the factor $(k_p/B)^2$. Using
(14) and the expression for the acoustic velocity in an electron gas at a
temperature T

$$v_o^2 = \frac{3kT}{m}$$

(89)

it follows that

$$\left(\frac{k_p}{B}\right)^2 = \frac{4\pi^2 m}{3k} \frac{f^2}{T B^2} \tag{90}$$

where m is the electronic mass, k the Boltzmann constant, and f the frequency of the incident electromagnetic radiation. For a frequency of 1GHz, a temperature of 2000°K, and B equal to unity, tne value of $(k_p/B)^2$ will be of the order 10^8. Consequently, all of the series will converge rather slowly ano require the calculation of many terms. Furthermore, these terms rapidly reach magnitudes which exceea the capabilities of most computers. While it is believed that this problem can be overcome by special programs, a lack of time prevented any efforts in this direction.

In order to obtain some idea of the behavior of the functions involved, $(k_p/B)^2$ was set equal to unity. Then for a frequency of 1GHz, B=50, and the thickness of the compressible inhomogeneous plasma layer equal to .014 meters, a reflection coefficient of about unity and a transmission coefficient effectively zero were observed for all angles of incidence between 5 and 85 degrees. It must be pointed out that the assumed numerical values actually describe a plasma at a temperature so high that it is no longer consistent with the governing equations and the assumptions behind them.
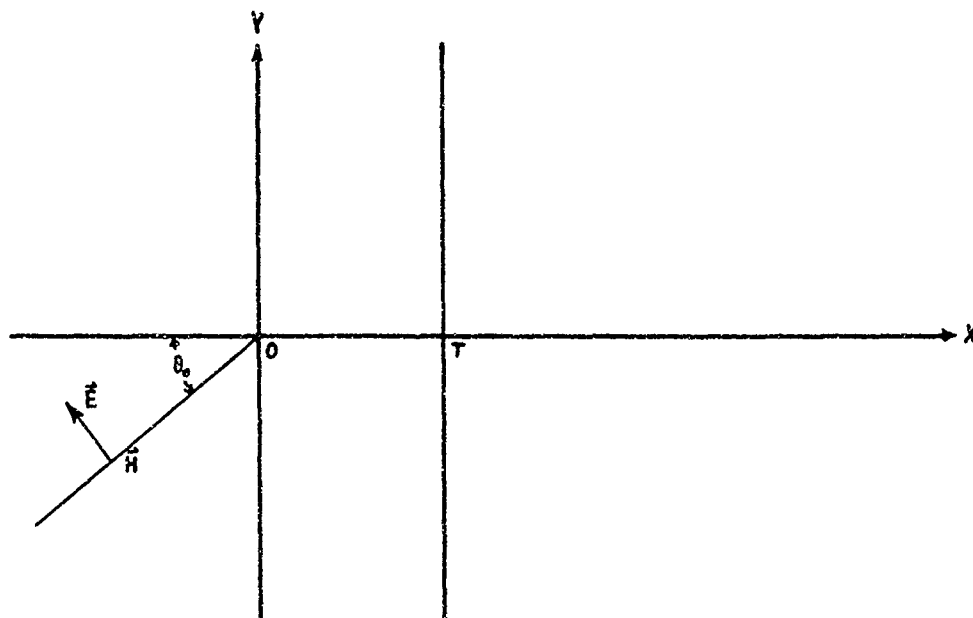
(i)
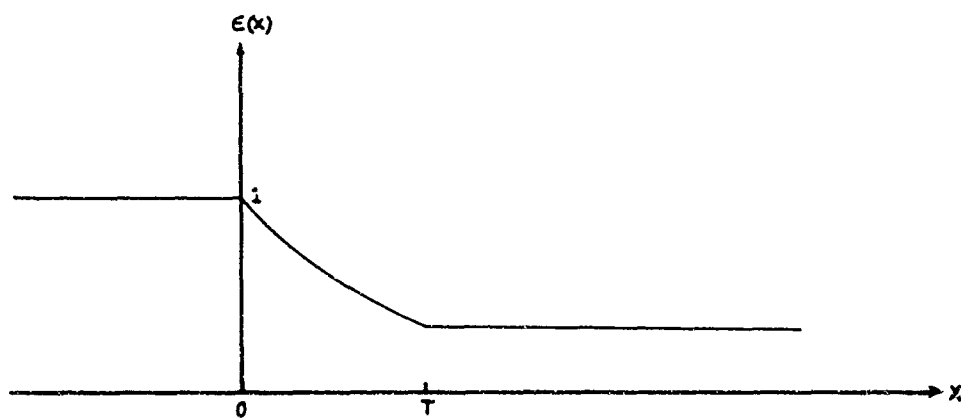
(ii)

Figure 1. (i) Plane wave incident obliquely upon a layer of
compressible inhomogeneous plasma occupying the region
$o \leq x \leq T$. (ii) The profile for all three regions.

REFERENCES

Cohen, M. H. (April 1962), Radiation in a plasma II, Equivalent sources, Phys. Rev., 126, 389-397.

Hessel, A., N. Marcuvitz, and J. Shmoys (January 1962), Scattering and guided waves at an interface between air and a compressible plasma, IRE Trans. Antennas and Propagation, AP-10, 48-54.

Rabenstein, A. L. (1966), Introduction to ordinary differential equations, (Academic Press, New York)

DOCUMENT CONTROL DATA - R&D

(Security classification of title body of abstract and indexing annotation must be entered when the overall report is classified,

1. ORIGINATING ACTIVITY (Corporate author)	2a. REPORT SECURITY CLASSIFICATION
Engineering Experiment Station The University of Arizona Tucson, Arizona	Unclassified
	2b GROUP

3. REPORT TITLE

PLANE WAVE PROPAGATION THROUGH A COMPRESSIBLE INHOMOGENEOUS PLASMA

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)

Scientific Report (Interim)

5. AUTHOR(S) (Last name, first name, initial)

Rahneberg, Eric
Tyras, George

6. REPORT DATE	7a TOTAL NO. OF PAGES	7b. NO. OF REFS
March 1967	27	3

8a. CONTRACT OR GRANT NO. AF 19(628)-3834	9a. ORIGINATOR'S REPORT NUMBER(S)
b. PROJECT NO. and Task No. 4642-02	Scientific Report No. 4
c. DOD Element No. 62405304	9b OTHER REPORT NO(S) (Any other numbers that may be assigned this report)
d. DOD Subelement No. 674642	AFCRL-67-0299

10. AVAILABILITY/LIMITATION NOTICES

Distribution of this document is unlimited.

11. SUPPLEMENTARY NOTES	12 SPONSORING MILITARY ACTIVITY
	Hq. AFCRL, (CRD) United States Air Force L. G. Hanscom Field, Bedford, Mass. 01730

13 ABSTRACT

The set of hydrodynamic equations together with Maxwell's equation which describe a nonuniform plasma as formulated by Cohen (1962) have been applied to a two dimensional problem where the magnetic field has only one component which is transverse to the direction of propagation and the profile is of the form e^{-Bx}. These assumptions lead to a set of second order coupled electromagnetic and hydrodynamic differential equations which describe the region under consideration. Solutions to this coupled set have been obtained by the application of the method of Frobenius. Considerable simplification resulted in the solutions when the assumption that the rms velocity of the electrons is much smaller than the speed of light was made. The resulting set of approximate solutions have been applied to the problem of obtaining reflection and transmission coefficients for the case of a plane wave incident from free space upon a layer of compressible inhomogeneous plasma.

DD FORM 1 JAN 64 **1473**

14		LINK A		LINK B		LINK C	
	KEY WORDS	ROLE	WT	ROLE	WT	ROLE	WT
Plane Wave Propagation Compressible Plasma Inhomogeneous Plasma							

INSTRUCTIONS

1. ORIGINATING ACTIVITY: Enter the name and address of the contractor, subcontractor, grantee, Department of Defense activity or other organization (*corporate author*) issuing the report.

2a. REPORT SECURITY CLASSIFICATION: Enter the overall security classification of the report. Indicate whether "Restricted Data" is included. Marking is to be in accordance with appropriate security regulations.

2b GROUP: Automatic downgrading is specified in DoD Directive 5200.10 and Armed Forces Industrial Manual. Enter the group number. Also, when applicable, show that optional markings have been used for Group 3 and Group 4 as authorized.

3. REPORT TITLE: Enter the complete report title in all capital letters. Titles in all cases should be unclassified. If a meaningful title cannot be selected without classification, show title classification in all capitals in parenthesis immediately following the title.

4. DESCRIPTIVE NOTES: If appropriate, enter the type of report, e.g., interim, progress, summary, annual, or final. Give the inclusive dates when a specific reporting period is covered.

5. AUTHOR(S): Enter the name(s) of author(s) as shown on or in the report. Enter last name, first name, middle initial. If military, show rank and branch of service. The name of the principal author is an absolute minimum requirement.

6. REPORT DATE: Enter the date of the report as day, month, year, or month, year. If more than one date appears on the report, use date of publication.

7a TOTAL NUMBER OF PAGES: The total page count should follow normal pagination procedures, i.e., enter the number of pages containing information.

7b. NUMBER OF REFERENCES: Enter the total number of references cited in the report.

8a CONTRACT OR GRANT NUMBER: If appropriate, enter the applicable number of the contract or grant under which the report was written.

8b, 8c, & 8d. PROJECT NUMBER: Enter the appropriate military department identification, such as project number, subproject number, system numbers, task number, etc.

9a. ORIGINATOR's REPORT NUMBER(S): Enter the official report number by which the document will be identified and controlled by the originating activity. This number must be unique to this report.

9b OTHER REPORT NUMBER(S): If the report has been assigned any other report numbers (*either by the originator or by the sponsor*), also enter this number(s).

10. AVAILABILITY/LIMITATION NOTICES: Enter any limitations on further dissemination of the report, other than those imposed by security classification, using standard statements such as:

(1) "Qualified requesters may obtain copies of this report from DDC."

(2) "Foreign announcement and dissemination of this report by DDC is not authorized."

(3) "U. S. Government agencies may obtain copies of this report directly from DDC. Other qualified DDC users shall request through

_____."

(4) "U. S. military agencies may obtain copies of this report directly from DDC. Other qualified users shall request through

_____."

(5) "All distribution of this report is controlled. Qualified DDC users shall request through

_____."

If the report has been furnished to the Office of Technical Services, Department of Commerce, for sale to the public, indicate this fact and enter the price, if known.

11. SUPPLEMENTARY NOTES: Use for additional explanatory notes.

12. SPONSORING MILITARY ACTIVITY: Enter the name of the departmental project office or laboratory sponsoring (*paying for*) the research and development. Include address.

13. ABSTRACT: Enter an abstract giving a brief and factual summary of the document indicative of the report, even though it may also appear elsewhere in the body of the technical report. If additional space is required, a continuation sheet shall be attached.

It is highly desirable that the abstract of classified reports be unclassified. Each paragraph of the abstract shall end with an indication of the military security classification of the information in the paragraph, represented as (TS), (S), (C), or (U).

There is no limitation on the length of the abstract. However, the suggested length is from 150 to 225 words.

14 KEY WORDS: Key words are technically meaningful terms or short phrases that characterize a report and may be used as index entries for cataloging the report. Key words must be selected so that no security classification is required. Identifiers, such as equipment model designation, trade name, military project code name, geographic location, may be used as key words but will be followed by an indication of technical context. The assignment of links, rules, and weights is optional.